RTG Elliptic PDEs Seminar

Seminar Details

- **Dates:** Feb 24-26 2025
- **Location:** Karlsruhe
- **Sources:** Taylor, Volume 1 [5] and Volume 2 [4], and Nicolaescu's "Lectures on the Geometry of Manifolds" [2]

Detailed plan

Talk 1 (Introduction and PDO's from an Algebraic Viewpoint) Briefly recall the definition of the Laplacian operator and the Laplace/Poisson equation (§2.4 of [5]); introduce the principal symbol of a differential operator as given in Taylor (§2.9 of [5]) give the definition of an elliptic operator, with Δ as an example (§3.9 of [5]). Goal is to motivate the seminar.
In the second part of your talk, cover Chapter 10.1 in [2]; define PDO's from an algebraic viewpoint. Cover the inductive definition of partial differential operators of order $\leq m$. Cover Ex. 10.1.2 in particular. Explain Lemma 10.1.3 (locality of PDO's). Show that the abstract definition reduces to operators of the form as in 10.1.5 (see also 10.1.17). Explain the definition of principal symbol (including well-definedness, see Lemma 10.1.8).[1] Define an *elliptic operator* (Def. 10.1.13). Discuss the Examples 10.1.18-10.1.20 (in decreasing precision).

Talk 2 (Sobolev Spaces) Follow [2], Chapters 10.2 and 10.3. Define Sobolev spaces on $\mathbb{R}^n$, the Sobolev norm and name its basic properties (10.2.10).[2] Maybe mention convolution and Lemma 10.2.13. State Sobolev's embedding theorem (10.2.21) without proof, Rellich-Kondrachov (10.2.24) and Morrey (10.2.25).[3] Define the L^p-space

[1] look at Section 8.1.3 of [2] if you have problems understanding the statements in terms of homogeneous polynomials

[2] you may sketch the proof very roughly if you have time

[3] If you still have time, you might want to sketch the proofs. For Morrey introduce Hölder spaces but maybe in a more classical formulation than Nicolaescu's.

and Sobolev space for sections of a bundle (10.2.30-10.2.33). You can skip 10.2.34 (or mention it orally). State 10.2.35 and 10.2.36 (independence of choice of metrics and connections if you have a compact base manifold). State 10.3.1 (interior elliptic estimate, this is important for elliptic regularity later). Cover elliptic regularity in more detail (this should be the heart of the talk): Cover 10.3.5 and state (no proof required) 10.3.6. Conclude from it the Weyl lemma (10.3.10). Maybe mention that the local results so far (on $\mathbb{R}^n$) lead to a global one on arbitrary compact manifolds (10.3.11). For additional details you might consult [3].

Talk 3 **(Application: Proof of Uniformization Theorem)** Cover all of Section §10.3.3 in [2] in detail. Lemma 10.3.3 provides a beautiful application. The goal is to sketch a proof of 10.3.15. Step 2 is just elliptic regularity (a nice application of the statements so far). For Step 1 sketch the minimizing idea (page 463) and Lemma 10.3.16. State and prove 10.3.20 and apply it to the modified functional $\widetilde{I}$ (state Lemma 10.3.21 but do not prove it!). Then the proof of 10.3.15 is finished. Briefly, deduce 10.3.24 from it and conclude the Uniformization Theorem (10.3.28).

Talk 4 **(Fredholm theory & Spectral Theory)** Work with [2] (also Appendix C.2 in Audin-Damian [1] is useful): Define what the adjoint of an (unbounded) operator is (10.4.1) and state 10.4.4. Now define what a Fredholm operator is (10.4.5) and present a sketch that the analytical realization L_a is Fredholm (10.4.7). Deduce the important Corollary 10.4.11. Sketch the argument that the index is continuous on the space of elliptic operators (10.4.13), so that you can conclude 10.4.16.
Lastly, as preparation for the next talks, review the spectral theory for compact and for self-adjoint Fredholm operators.

Talk 5 **(Schwartz kernels and Mehler's Formula)** Work with Taylor [4]. According to Atkinson's theorem Fredholm operators are 'almost invertible' in the sense that they admit parametrices. Following Chapter 7.2, introduce Schwartz kernels as an explicit realization of such parametrices. As an example mention Mehler's formula, see Chapter 8.6 (6.34), and discuss asymptotic expansions of the 'heat kernel', see Chapter 7.13.

Talk 6 **(Hodge Theory)** Work with Nicolaescu [2]. Briefly recall Theorem 10.4.19 and illustrate it with Example 10.4.20. Give the definition of an *elliptic complex* (10.4.27) and take 10.4.28 as the decisive example. State the Hodge theorem (10.4.29) and construct the isomorphism (which is the descent of an orthogonal projection), i.e.

prove the theorem briefly. Conclude the classical Hodge theorem (10.4.31) that any cohomology class can be represented by a unique harmonic form. Finish your talk by giving the one-line proof that the Hodge star $*$ is Poincaré duality via the natural identifications (10.4.33).

Talk 7 **(Clifford algebras, spinor representations and Weitzenböck formulas)** Cover Chapter 10, Section 1-4, in [4]

- Working on a Riemannian manifold (M, g) argue that there is a fibrewise algebra homomorphism $\theta : Cl(M, g) \longrightarrow End(\Lambda^* M)$ and that a metric connection ∇ on TM extends to a Clifford connection on $\Lambda^* M$.
- Identify the associated Dirac operator as $d + d^*$ and mention that the Hodge decomposition on differential forms gives an elementary way to see that $Ind(d + d^*) = \chi(M)$.
- Point out that whenever a Dirac operator $D : \Gamma(M, E) \circlearrowleft$ is derived from a Clifford connection, the difference $D^2 - \nabla^*\nabla$ can be expressed in terms of the curvature tensor $\mathcal{K}(E, \nabla)$. Going back to the example $E = \Lambda^* M$, conclude that compact Riemannian manifolds with positive Ricci tensor (like e.g. certain Einstein manifolds) do not admit any non-zero harmonic 1-forms.
- Define the groups $Spin, Pin \subset Cl(V, g)$ and mention the double cover homomorphism $Spin \longrightarrow SO(V, g)$. Use orthonormal frames ('vielbeins') to construct the spinor bundles $S(\tilde{P}) \longrightarrow M$. Focus on the case of even dimensional manifolds and decide whether or not to mention Stiefel-Whitney classes. Conclude by stating the Weitzenböck formula for spinor bundles, which instead of the Ricci tensor now involves the scalar curvature of M.

Talk 8 **(Atiyah-Singer Index Theorem)** Chapter 10, Section 5-7, in [4]

- Given a first order elliptic operator D use spectral theory to reformulate the index as a well-defined 'heat trace'

$$Ind(D) = Tr(e^{-tD^*D}) - Tr(e^{-tDD^*}) \qquad (1)$$

  and invoke asymptotic expansions of the respective integral kernels to express the r.h.s. in terms of finite-dimensional traces of the coefficients.
- If D is a Dirac operator, analyze the grading $\mathbb{C}l^{(2j)}(2k) \otimes F$ of the asymptotic coefficients and use your findings to reexpress

the r.h.s. of (1) in terms of the differential forms $Ch(F)$ and $\hat{A}(M)$, thereby proving the Atiyah-Singer index formula Thm 5.1.

- (If you have time) Returning to the Dirac operator $D = d + d^*$ from the previous talk, derive the Chern-Gauss-Bonnet formula Thm 7.2, which expresses $\chi(M)$ in terms of the curvature Pfaffian.

Talk 9 (Optional)
(***Spin***^{c}**-structures and the Riemann-Roch Theorem)**
(Ch.10 sections 8-9 in [4]):

- Discuss the relation between *spin*- and *spin*^{c}-structures.
- Explain the index formula Thm 8.1 and how the 'clutching construction' can be used to compute the index of arbitrary elliptic operators.
- Working on a compact Riemann surface define the Cauchy-Riemann operator of a line bundle and interpret its index in terms of holomorphic sections.
- Argue that Atiyah-Singer implies the Riemann-Roch formula Thm 9.1 and discuss its relation to meromorphic sections. You may find it interesting to also include Exercise 10.9.5 about quadratic differentials.

References

[1] Michele Audin, Mihai Damian, and Reinie Erné. *Morse theory and Floer homology.* Vol. 2. Springer, 2014.

[2] Liviu Nicolaescu. *Lectures on the Geometry of Manifolds.* Sept. 2007. ISBN: 978-981-277-862-8; 981-277-862-4. DOI: `10.1142/6528`.

[3] John J. F. Fournier Robert A. Adams. *Sobolev Spaces.* Elsevier, 2003.

[4] M.E. Taylor. *Partial Differential Equations II: Qualitative Studies of Linear Equations.* Applied Mathematical Sciences. Springer New York, 2010. ISBN: 9781441970527. URL: `https://books.google.de/books?id=scOX7_JUp_kC`.

[5] Michael E. Taylor. *Partial Differential Equations I: Basic Theory.* 2nd ed. Vol. 115. Applied Mathematical Sciences. Springer New York, NY, 2011. ISBN: 978-1-4419-7055-8. DOI: `10.1007/978-1-4419-7055-8`.